A Invertible Dimension Reduction of Curves on a Manifold

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Abstract

In this paper, we propose a novel lower dimensional representation of a shape sequence. The proposed dimension reduction is invertible and computationally more efficient in comparison to other related works. Theoretically, the differential geometry tools such as moving frame and parallel transportation are successfully adapted into the dimension reduction problem of high dimensional curves. Intuitively, instead of searching for a global flat subspace for curve embedding, we deployed a sequence of local flat subspaces adaptive to the geometry of both of the curve and the manifold it lies on. In practice, the experimental results of the dimension reduction and reconstruction algorithms well illustrate the advantages of the proposed theoretical innovation.

1 Introduction

For many pattern recognition problems, the feature space is nonlinear and of high dimensionality and nonlinearity, for example, face recognition, image classification and shape recognition. Many dimension reduction techniques [1] are there-

fore naturally introduced to ease the modeling, computation and visualization of the feature space. Dimension reduction has been widely used with different vision techniques such as contour tracking [3], face recognition [4] and object recognition [2].

One of the key points of dimension reduction is how to represent the geometry of the original space in a lower dimensional space [9, 10]. Most of the previous dimension reduction research [1] focuses on embedding the points in the original space into a lower dimensional space. Beyond the point-wise dimension reduction, the dimension reduction for a sequence of points or an evolving curve in the original feature space is more important in applications where features of interest are time varying such as, face expression [4], events indexing in video sequence as a shape process [11] and human activity as a shape sequence [3]. To the best of our knowledge, only a few works [3, 2] address the problem of how to represent a curve in the original feature space in a lower dimensional space.

The dimension reduction of high dimensional curves may be categorized into two classes. One is to reduce the whole curve space to a lower dimension as in [3, 12, 13]. The other one is to reduce a particular the curve to a corresponding lower dimensional curve as in [2]. The drawback of the first one is the Euclidean assumption by using an approximation with a spline function. Moreover, as in [3] by directly applying PCA to the spline coefficients, the global dimension reduction does not consider the intrinsic dimensionality of a particular high dimensional curve. A curve sequence like walking, for example, is much simpler than a dancing sequence. Thus the dynamics of walking might be represented in a much lower dimensional space in comparison to other complex activities. The second category of dimension reduction as in [2], considered the geometry of a particular curve in the original space, which, due to the chosen implementation of the Whitney embedding theorem, is computationally heavy on account of the iterative subspace search. At the same time, there exists no invertible mapping from the embedding space to the original one, thus limiting its extension to a generative process.

Motivated by the importance of a lower dimensional representation of a manifold-valued curve and the limitations of the current techniques, in this paper a novel dimension reduction of a shape manifold valued curve is proposed. In contrast with previous curve dimension reduction techniques [3, 2], the proposed method has the following advantages:

1. the dimension reduction is adaptive to the Riemannian geometry of a shape space.

2. the computation is linear in the length of the curve times the dimension of the shape space.

3. with a proper dimensionality, there exists a reconstruction as a subspace parallel transport on a manifold.

To introduce the idea of the proposed dimension reduction framework for the manifold valued curves, this paper follows the moving frame formulation in [6, 7]. According to [7, 6], every curve on a manifold may be represented by a differential equation,

$$X_t = X_0 + \int_0^t \sum_{i=1}^{dim(M)} V_i(X_t) dZ_i(t) \tag{1}$$

where, $V(X_t)$ is a vector field on a manifold, and Z_t is a curve in a Euclidean space. Under such representation, to introduce a lower dimensional representation of X_t, a method is proposed to find the optimal vector field $V(X_t)$ such that:

1. The sequence of $V(X_t)$ is path dependent and uniquely determined by X_t and initial condition $V(X_0)$ by a efficient computation.

2. The resulting dZ_t from X_t and $V(X_t)$ have a distribution that lies in a lower dimensional subspace.

The first requirement on $V(X_t)$ means that X_t can be represented by only the initial condition $V(X_0), X_0$ and the embedded curve Z_t. According to [6] such property yields the 1-1 correspondence between X_t and Z_t. The second requirement is more intuitive that among all the $V(X_t)$ satisfied the first requirement, the optimal one is selected such that the variance of tangents is best represented in a subspace.

The first requirement is satisfied by select $V(X_t)$ that is a horizontal vector field [7] defined by the Levi-Civita connection as in Section 2.2 with a L^2 metric induced to the shape manifold M from the ambient space. According to [6, 7], under a defined connection, for $X_t \in M$ and initial condition $V(X_0)$, there exists a unique selection of moving frame $V(X_t)$. Thus given $V(X_0)$ the resulting curve in Euclidean space have one-one correspondence to X_t. The reason of using the Levi-Civita connection have two folds. The first one is that it is a non-flat connection. In contrast to the other category: flat connection, it is path dependent. Such property is important for dimension reduction because it provides more adaptivity to the geometry of the path on the manifold. The second reason is that in comparison with the general nonflat connections defined by a PDE on manifold, such

particular Levi-Civita connection in Section 2.2 is consistent with the metric of the shape manifold in [5] that this paper is based on, and is computationally more efficient for a known ambient space and normal space of the manifold, as is the case of our shape manifold.

The second requirement is achieved by the subspace representation resulting from maximizing the variance of the corresponding curve development $dZ(t)$ with different choices of $V(X_0)$. In theory, without knowing the initial condition $V(X_0)$ the Levi-Civita connection only determines the vector field $V(X_t)$ up to a group action within the fiber of the Frame bundle (the total space of frames of the tangent space of a manifold) [7]. Among different choices for $V(X_t)$ along the fiber in the frame bundle, a optimal frame is selected such that the previously discussed second requirement is satisfied to make the resulted Zt better representable in a lower dimensional space. A $\tilde{V}(X_0) = \{x_0, e_1, e_2, \cdots, e_{dim(M)}\}$ is selected by carried out PCA of all the parallel transported tangents along X_t.let $\{e_1, e_2, \cdots, e_L\}$ are the eigenvectors corresponding to the L largest Eigenvalues in the PCA calculation. Then A subspace spanned by the subset $\{e_1, e_2, \cdots, e_L\}$ is constructed to better represent the variance of all the tangents along X_t in a lower dimensional space.

To summarize, a curve in the shape manifold M is represented as

$$X_t = X_0 + \int_0^t \sum_{i=1}^{L} \tilde{V}_i(X_s)d\tilde{W}_i(s) \tag{2}$$

where $\tilde{V}_i(X_t)$ can be represented by L curves in R^3 and $\tilde{Z}_i(t)$ is represented by a curve in R^L. Our proposed dimensional reduction achieves the following points,

1. a lower dimensional representation of curves on a manifold.

2. The moving frame representation can be generated to a stochastic representation of random process on a manifold. Thus the propose method can be well adapted to a generative modeling in a lower dimensional space.

3. the computation is linear in the dimension of X_t.

4. there exists a reconstruction from the lower dimensional representation to the original curve on the shape manifold.

The rest of the paper is organized as following. Section 2 provide necessary background knowledge of shape manifolds and the geometric calculation on it. In Section 3 and 4, the dimension reduction framework is introduced, which includes the curve embedding and the curve reconstruction.

2 Preliminary

In this section, we first introduce in Subsection 2.1 the shape manifold that the dimension reduction is based on. Then in Subsection 2.2 a fundamental moving frame representation of curves on manifold is briefly introduced.

2.1 Shape Manifold

According to [5], a planar shape is a simple and closed curve in $\mathbb{R}^2$,

$$\alpha(s) : I \to \mathbb{R}^2, \tag{3}$$

where an arc-length parameterization is adopted. A shape is represented by a *direction index function* $\theta(t)$. With such a parameterization, $\theta(s)$ may be associated to the shape by

$$\frac{\partial \alpha}{\partial s} = e^{j\theta(s)}. \tag{4}$$

The ambient space of the manifold of θ is an affine space based on $\mathbb{L}^2$. Thus

$$\theta \in A(\mathbb{L}^2). \tag{5}$$

The restriction of a shape is that it must be a closed and simple curve, and invariant over rigid Euclidean transformations. The shape manifold M is defined by a level function ϕ as

$$\phi(\theta) = \left(\int_0^{2\pi} \theta ds, \int_0^{2\pi} \cos(\theta)ds, \int_0^{2\pi} \sin(\theta)ds \right). \tag{6}$$

$$M = \phi^{-1}(\pi, 0, 0). \tag{7}$$

One of the most important properties of M is that the tangent space TM is well defined. Such a property not only simplifies the analysis, but also makes the incremental computation possible,

$$T_\theta M = \{ f \in \mathbb{L}^2 | f \perp span\{1, \cos(\theta), \sin(\theta)\} \}. \tag{8}$$

In addition, an iterative projection is proposed in [5] to project the point in ambient space back to the shape manifold M. The idea is that each time θ is updated as $\theta + d\theta$, where $d\theta$ is orthogonal to the level set $\phi^{-1}(\phi(\theta))$. The $d\theta$ is

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calculated as $d\phi^{-1}((\pi, 0, 0) - \phi(\theta))$. For the detailed form of the Jacobian of $d\phi$, one could refer to [5].

The problem of this manifold for our stochastic modeling is the infinite dimension. The mapping of random process in [6] from a manifold to a flat space is only defined on a finite dimensional manifold. Therefore, a Fourier approximation of the shape manifold we discussed above is developed, such that the dimension is reduced to a finite number.

2.2 Moving Frame Geometry

Let X_t be a curve on a manifold M, then following the moving frame representation [8], the tangent dX_t of manifold valued curve X_t may be written as:

$$dX_t = \sum_i V_i(X_t)dZ_t, \tag{9}$$

where $\{V_i(X_t)\}_{i=1,2,\cdots,dim(M)}$ is a frame of the tangent space at X_t, which is denoted as $T_{X_t}M$. Consequently dZ_t may be understood as a linear coefficient of dX_t under the representation of $V(X_t)$. Such moving frame representation is widely used in geometry studies of curves. For example in the Frenet theorem [8] the geometry of curve is uniquely identified by the curvature of $V(X_t)$ up to a rigid Euclidean transformation. Recently in computer graphics, a rotation minimizing moving frame [17] are developed to avoid the singularity of original moving frame used in Frenet theorem.

In this paper, the moving frame representation is utilized for the similar purpose of representing a curve on a manifold. In contrast to the previous designs of moving frame, in this paper we elaborate to propose a adaptive moving frame to represent a high dimensional curve in a lower dimensional space. Vector field $V(X(t))$ are developed as a sequence of parallel frames along X_t. The parallelism is defined under a Levi-Civita connection. The advantages of such innovation are briefly introduced in the following:

1. Once $V(X_t)$ are parallel according to the curve development theory in [6], X_t can be represented uniquely as (X_0, V_0, Z_t). Such property yields the invertibility of the proposed dimension reduction of X_t. In other words, by the representative curve Z_t in a lower dimensional space, X_t can be reconstructed when the initial conditions (X_0, V_0) are available.

2. Under Levi-Civita connection, the parallel frames are calculated according to metric of the manifold M. Such property implies that the shape of Z_t will

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reflect well the shape of X_t on M. For example, if X_t is a geodesic curve, then in the proposed method Z_t will be a straight line in Euclidean space

3. The particular shape manifold [5] we adopted in this paper have a known tangent and normal space in the ambient space which greatly simplified the calculation of the moving frames under Levi-Civita connection.

The Levi-Civita connection may be viewed as the Christoffel Symbol Γ that defines the directional derivative in M.

$$D_{\partial x_i} \partial x_j = \sum_k \Gamma_{i,j}^k \partial x_k \tag{10}$$

where x_i is the coordinate function of an arbitrary point m in M. The tangents of M can be written as linear combination of $\{\partial x_i\}_{i=1,2,\cdots,dim(M)}$. For example at X_0, $V_i(X_0) = \sum_j a_j^i \partial x_j$ and $dX_0 = \sum_j b_j \partial x_j$. Thus according to Equation (10), we can analytically calculate the derivative of $V_i(X_t)$ along X_t in direction dX_t, which is denoted as $D_{\frac{\partial X_t}{\partial t}}(V_i(X_t))$. $V(X(t))$ is parallel if $\forall i$,

$$D_{\frac{\partial X_t}{\partial t}}(V_i(X_t)) = 0. \tag{11}$$

However in practical problems, it is usually impossible to chart the manifold. In other words, the coordinate functions x of M are usually unknown. In this paper, the shape manifold [5] we introduced in Section 2.1 have some nice properties that allows us to calculate $D_{\frac{\partial X_t}{\partial t}}(V_i(X_t))$ in terms of a projection of Euclidean calculus in the ambient space onto the tangent space of the manifold M.

$$D_{\frac{\partial X_t}{\partial t}}(V_i(X_t)) = Proj_{T_{X_t}M}\left(\frac{\partial \bar{V}_i(X_t)}{\partial t}\right), \tag{12}$$

where $\bar{V}_i$ is the vector representing V_i in the ambient space of M. Since the ambient space is an affine space based on $\mathbb{L}^2$, thus $\bar{V}_i$ is calculable as a real vector in $\mathbb{L}^2$. The $Proj_{T_{X_t}M}$ is a mapping of vectors in $\mathbb{L}^2$ onto the tangent space of M.

3 Dimension Reduction

The dynamics of some common human activities such as walking and running, are relatively simpler in comparison to a complex activity like dancing. Thus it is nature to speculate that the simple activity could be represented in a lower dimensional subspace on the shape manifold. To learn such sub-manifold, a dimension

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reduction is proposed for a curve on a shape manifold. As described in the previous section, first the horizontal vector field $V(X_t)$ is defined by a Levi-Civita connection. Then Principle Component Analysis (PCA) is carried out to compute the optimal vector field $\tilde{V}(X_t) = \{x_t, e_1, e_2, \cdots, e_L\}$ such that all the parallel transported tangents along X_t could be represented in a lower dimensional space.

The shape sequence on a manifold M can be represented by a differential equation as,

$$X_t = X_0 + \int_0^t \sum_{i=1}^{dim(M)} V_i(X_t)dZ_i(t), \tag{13}$$

where, $V(X_t)$ is selected to be the horizontal vector field on a manifold and Z_t is the curve development in a Euclidean space. Such a formulation have the property according to [7, 6] that the resulting Z_t is 1-1 corresponded to X_t with the initial condition $V(X_0), X_0$.

The idea of the proposed dimension reduction is to reduce the dimension for a manifold valued curve X_t by learning a sequence of subspaces $\tilde{V}(X_t)$ on the shape manifold M, such that the resulting dZ_t have a distribution that concentrate in a lower dimensional subspace.

A Levi-Civita connection is adapted on the shape manifold according to the L^2 metric induce from the ambient space. According to the definition of covariant derivative under Levi-Civita connection as in Section 2.2, the horizontal vector field $V(X_t)$ is the solution of the following differential equation.

$$\forall i, D_{\frac{\partial X_t}{\partial t}}(V_i(X_t)) = 0, \tag{14}$$

with initial condition $(X_0, V(X_0))$.

The subspace moving frames $\tilde{V}(X_t)$ for dimension reduction is constructed by select the optimal initial condition $V(X_0) = \tilde{V}(X_0)$. First parallel transport all the tangents $\frac{\partial X_t}{t}$ to the tangent space at X_0. With the moving frame formulation in Equation (13), the parallel transportation results is a set τ

$$\tau = \{ \sum_{i=1}^{dim(M)} V_i(X_t)dZ_i(t)\}_{t=1,2,\cdots,N} \tag{15}$$

According to [7], the set τ is invariant to the choice of $V(X_0)$. Thus in the first step an arbitrary initial condition is assigned.

Then Let the $\{e_1, e_2, e_3, \cdots, e_L\}$ be the eigenvectors that corresponding to the L largest eigenvalues of vectors in τ. The optimal initial condition is constructed

as

$$\tilde{V}_i(X_0) = e_i. \tag{16}$$

The solution $\tilde{V}(X_t)$ of Equation (14) with initial condition $(X_0, \tilde{V}(X_0))$ is the moving frames we used to represent the X_t by the corresponding $\tilde{Z}_t$ in a lower dimensional space R^L

In the following, we provide the solution to the differential equation (14). According to Equation (12) and normal space expression of the shape manifold M in Section 2.1, the above equation can be solved as the following. Let $\{B_1(t), B_2(t), B3(t)\}$ be the orthogonalization of $\{1, cos(X_t), sin(X_t)\}$, which is the basis of normal space of the shape manifold M. $\forall i$,

$$\frac{\partial V_i(X_t)}{\partial t} = a_1 B_1(t_0) + a_2 B_2(t_0) + a_3 B_3(t_0). \tag{17}$$

In the numerical calculation of the Euclidean derivatives $\left(\frac{V_i(X_t)}{\partial t}\right)$ are implemented as follows. For a small enough $h > 0$,

$$\left(\frac{\partial V_i(X_t)}{\partial t}\right) = \frac{V_i(X_{t+h}) - V_i(X_t)}{h}. \tag{18}$$

Substitute the Equation (18) into the covariant derivative Equation (17), $V_i(X_{t+h})$ could be written as,

$$V_i(X_{t+h}) = V_i(X_t) + a_1 B_1(t_0) + a_2 B_2(t_0) + a_3 B_3(t_0). \tag{19}$$

Since $V_i(X_{t+h}) \in T_{X_{t+h}} M$, the parameter a_i should satisfy that,

$$< V_i(X_t) + a_1 B_1(t_0) + a_2 B_2(t_0) + a_3 B_3(t_0), B_1(t_0 + h)) >= 0,$$

$$< V_i(X_t) + a_1 B_1(t_0) + a_2 B_2(t_0) + a_3 B_3(t_0), B_2(t_0 + h) >= 0,$$

and

$$< V_i(X_t) + a_1 B_1(t_0) + a_2 B_2(t_0) + a_3 B_3(t_0), B_3(t_0 + h) >= 0.$$

From the above three equations, we have,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = -A^{-1}V, \tag{20}$$

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where $A =$

$$\begin{pmatrix} B_1(X_t)B_1(X_{t+h}) & B_2(X_t)B_1(X_{t+h}) & B_3(X_t)B_1(X_{t+h}) \\ B_1(X_t)B_2(X_{t+h}) & B_2(X_t)B_2(X_{t+h}) & B_3(X_t)B_2(X_{t+h}) \\ B_1(X_t)B_3(X_{t+h}) & B_2(X_t)B_3(X_{t+h}) & B_3(X_t)B_3(X_{t+h}) \end{pmatrix}.$$

So the parameter a_i could be solved in a linear fashion. Thus the shape sequence can be represented as follows,

$$X_t = X_0 + \int_0^t \sum_{i=1}^{L} \tilde{V}_i(X_t) d\tilde{Z}_i(t), \qquad (21)$$

where $d\tilde{Z}_i(s) = <\frac{\partial X_t}{\partial t}, \tilde{V}_i(X_t)>$ is a curve development in R^L.

Figure 2 illustrates the R^3 embedding of the shape sequence in Figure 1. The curvature of the embedded curve in R^3 in Figure 2 represent well the change of original shape sequence along time.

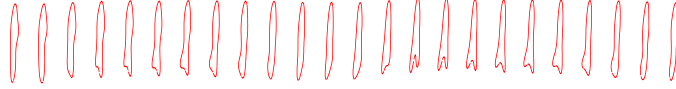A original shape sequence on the shape manifold

Figure 1: The original shape sequence for activity: Running

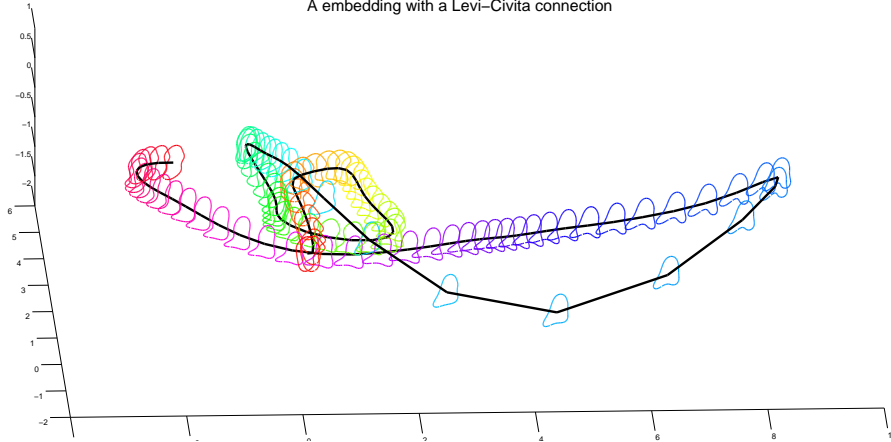A embedding with a Levi–Civita connection

Figure 2: The embedding of a shape sequence in Figure 1 into R^3

In the following, the additional different activities from database [16] is considered, such as Jumping, Siding, Skipping and Waving. The dimension reduction result in Figures 3,5,4,6, also well visualized the evolution on the shape manifold M.
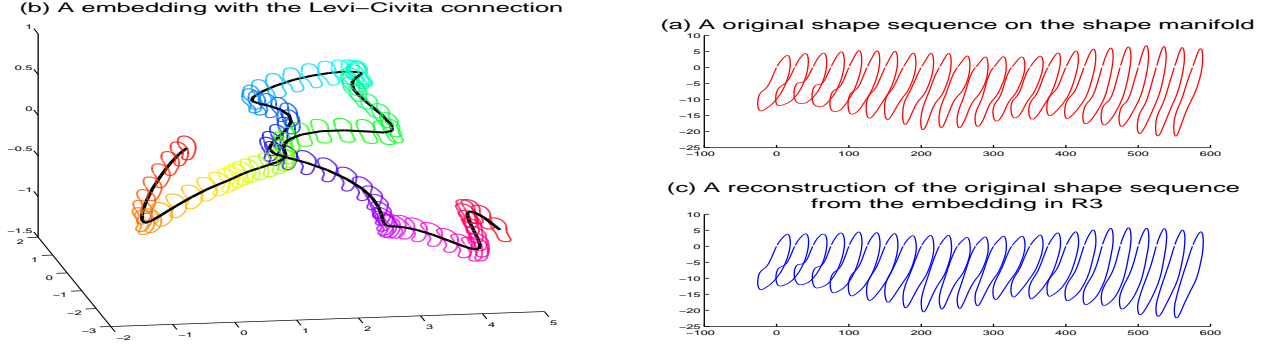

Figure 3: (a) The original shape sequence; (b) The embedding into R^3; (c) The reconstruction of the shape sequence from the embedding curve in R^3.
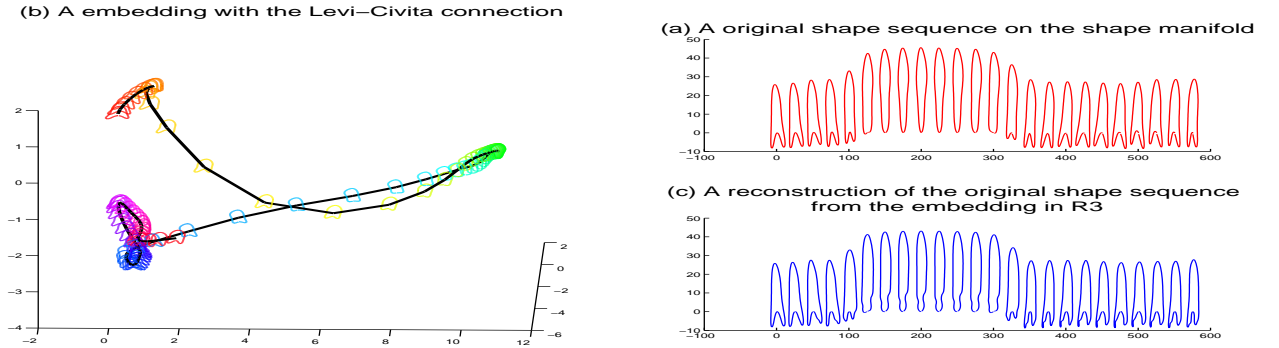

Figure 4: (a) A original shape sequence; (b) The embedding into R^3; (c) the reconstruction of the shape sequence from its corresponding embedding curve in R^3.
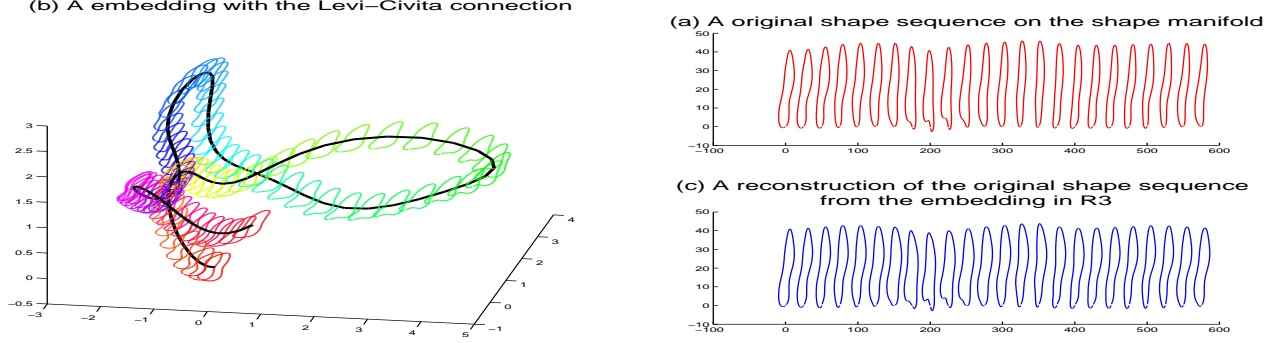
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Figure 5: (a) A original shape sequence; (b) The embedding into R^3; (c) The reconstruction of the shape sequence from its corresponding embedding curve in R^3.

4   Reconstruction from the Lower Dimensional Representation

As introduced in the previous sections, one of the claimed advantages of the proposed dimension reduction is that it is possible to generate the original shape sequence from the dimension reduction results. Such an invertible property makes our lower dimensional shape process embedding a compelling way to learn the generative model of the shape sequence.

Let $\tilde{Z}(t)$ be the embedded shape process in R^L. A approximation of the original sequence on the shape manifold M may be generated by the following differential equation:

$$X_t = X_0 + \int_0^t \sum_{i=1}^{L} \tilde{V}_i(X_0) d\tilde{Z}_i(t) \tag{22}$$

where the horizontal vector field $\tilde{V}_i(X_t))$ is generated uniquely by the differential equation 14 with the initial frame $X_0, \tilde{V}_i(X_0)$.

Numerically Equation (22) is implemented as a difference function,

$$X_k = f(X_{k-1}) + \sum_{i=1}^{L} \tilde{V}_i(X_0) d\tilde{Z}_i(k-1)) \tag{23}$$
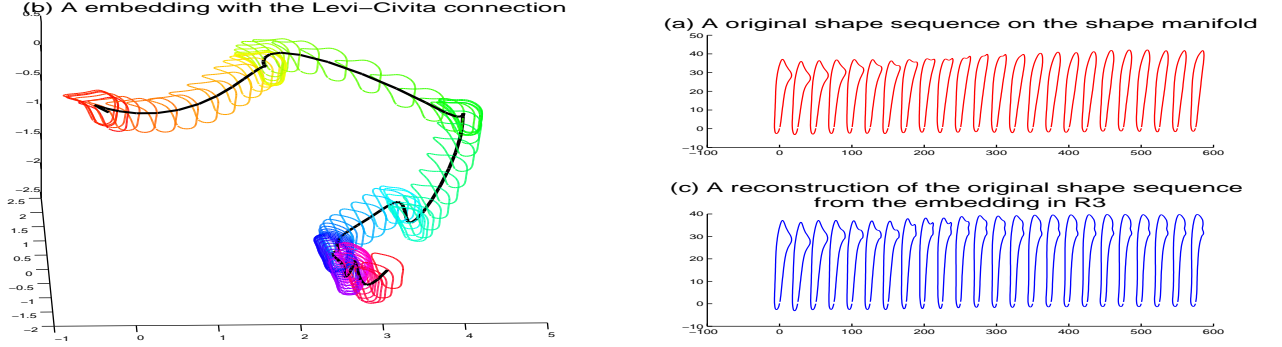
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Figure 6: (a) A original shape sequence; (b) The embedding into R^3; (c) The reconstruction of the shape sequence from its corresponding embedding curve in R^3.

where f is the iterative projection from the ambient space to the shape manifold as in [5], which is also briefly introduced in Section 2.1.

Figure 7 illustrates the reconstruction result of the shape sequence of activity: running. The reconstruction is from the embedding curve $\tilde{Z}(t) \in R^3$ and the initial condition X_0, $\tilde{V}_i(X_0)$ to the shape manifold M.

The Figure 7 demonstrates that the reconstruction approximates well the original shape sequence. More reconstruction results of the shape sequence of other human activities are shown in Figures 3,5,4,6.

5 Conclusion

Dimension reduction of points on a high dimensional manifold is well studied in the past decades. Only a few works, however, address the problem of dimension reduction of curves on a manifold. In this paper a novel dimension reduction technology is proposed for shape dynamics with the following advantages that, as far as we know, no previous work has ever achieved:

1. The subspace is learned nonlinearly according to the geometry of the underlying manifold.

2. The computation is linear in the size of data.

3. There exists an efficient reconstruction from the lower dimensional representation to the original curve on a shape manifold.

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(a) A original shape sequence on the shape manifold

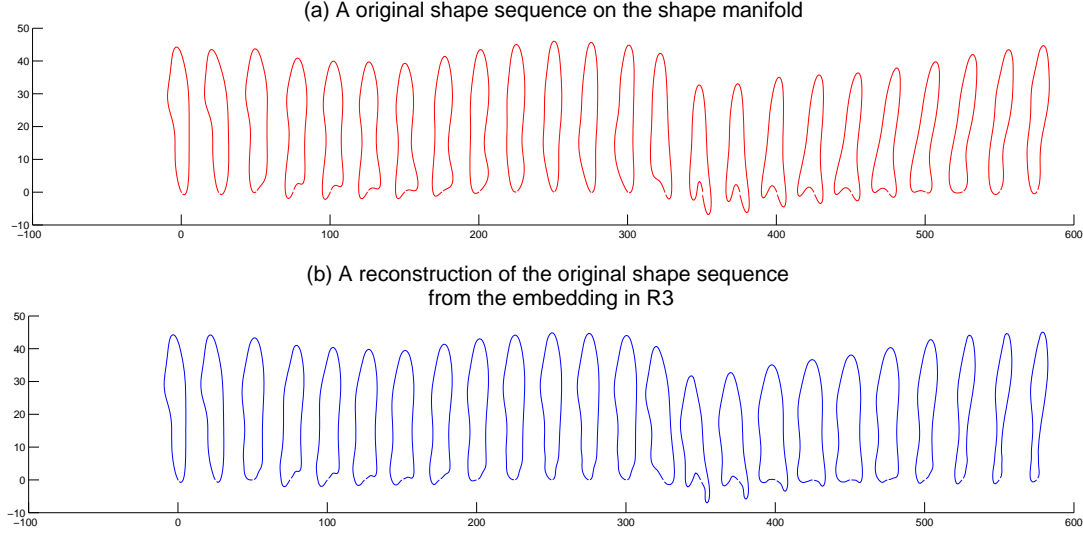(b) A reconstruction of the original shape sequence from the embedding in R3

Figure 7: The reconstruction result for original sequence in Figure 1 with embedding result in Figure 2

4. The proposed dimension reduction technique provides both analytical and practical foundations for generative modeling of shape dynamics.

This work also first apply the differential geometry tools such as moving frame representation and Levi-Civita connection to dimension reduction of curves on a manifold. It provides a new perspective on the dimension reduction with a moving frame formulation to naturally characterize the nonlinear features of a manifold valued dynamics in a linear subspace.

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